Research report 151

ON PERFECT PAIRS OF TREES IN A GRAPH

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§1 Introduction

In many problems, especially those concerned with lumped circuits, pairs of (spanning) trees play an important role. Most "geometrical" properties of the set of all trees are related to the concept of distance between trees [1]. In that context several notions are introduced, in particular those of central tree [2], extremal tree [3] and maximally distant pairs of trees [3]. Some deep graph-theoretical concepts are related to the investigation of distances between trees, for example the concept of a principal partition of a graph [3] and that of topological degree of freedom [4]. Several important applications are also based on these notions [7]-[12].

Since Kishi and Kajitani realised the importance of introducing the concept of maximally distant trees [3] and Ohtsuki, Ishizaki and Watanabe promoted the concept of topological degree of freedom [4], many papers have been published which more closely characterise those properties of graphs connected with the space of trees [13]-[20]. The appearence of new concepts can be a spurious process and it may not at first be clear which are of value and whether the most useful definitions have been made. In this paper we introduce a new concept, that of a **perfect pair of trees** of a graph. We shall be particularly concerned with closely characterising the properties of this concept. We hope that our concept of a perfect pair of trees provides excellent intuitive insight within this area of study. Throughout this paper we shall be concerned only with 2-connected graphs.

§2 Preliminaries

This section is devoted to some definitions and assertions related to material that follows. We presume that the reader is familiar with the following basic notions in graph theory : graph, edge, circuit and cutset. We take these to be primary notions that need not be defined. However we will define all other notions on the basis of these. Throughout we denote a graph by G and its edge set by E. The terms circuit, cutset, tree, cotree, forest and coforest will be used here to mean a subset

The following assertion plays a central role in this paper.

Assertion 6

The following five statements are equivalent:

i) t_2 is maximally distant from t_1

ii) the fundamental circuit with respect to t_2 defined by an edge in $t_1^* \cap t_2^*$ contains no edges in $t_1 \cap t_2$.

iii) the fundamental cutset with respect to t_1^* defined by an edge in $t_1 \cap t_2$ contains no edges in $t_1^* \cap t_2^*$.

iv) $| t_1 \backslash t_2 | = \text{rank } t_1^*$.

v) the number of edges in $t_1 \cap t_2$ is equal to the maximal number of independent cutsets of the graph that belong entirely to the tree t_1.

Proof

i) $\Rightarrow$ ii)

Suppose that a fundamental circuit with respect to t_2, defined by an edge $c \in t_1^* \cap t_2^*$, contains an edge $b \in t_1 \cap t_2$, then $t'_2 = (t_2 \backslash \{b\}) \cup \{c\}$ is a tree. Due to $|t_1 \backslash t'_2| = |t_1 \backslash t_2| + 1$, we conclude that the distance between t_1 and t'_2 is bigger than between t_1 and t_2, which contradicts the assumption that t_2 is maximally distant from t_1.

ii) $\Leftrightarrow$ iii)
This is evident from the following well known statement: two edges belong to a circuit iff they both belong to the same cutset. Notice that in the present case, both circuit and cutset are in fact subsets of $t_1 \backslash t_2$.

ii) $\Rightarrow$ iv)

If condition ii) is satisfied then fundamental circuits with respect to t_2, defined by edges in $t_1^* \cap t_2^*$ contains only edges from $t_2 \backslash t_1$. Hence rank $t_1^* \leq |t_2 \backslash t_1|$. On the other hand, for any pair of trees (t_1, t_2) rank $t_1^* \geq |t_2 \backslash t_1|$. (see Assertion 2, part i)). Therefore rank $t_1^* = |t_2 \backslash t_1| = |t_1 \backslash t_2|$.

iv) $\Rightarrow$ i)

Suppose that the condition iv) is satisfied, and let t is an arbitrary tree. So, the following relations hold:

$$|t_1 \backslash t_2| = \text{rank } t_1^* \text{ (Condition iv))}$$

$$|t_1 \backslash t\,| \leq \text{rank } t_1^* \text{ (Assertion 2, part i))}$$

Consequently, for any tree t, $|t_1 \backslash t| \leq |t_1 \backslash t_2|$.

(iv) $\Leftrightarrow$ (v)

Let c be a number of vertices of the graph G. Also let p(t) be the number of connected components of the graph obtained by removing all edges from t, and let k(t) be the maximal number of independent cutsets in t_1. Then the following relations are always valid:

1) $\text{rank } t^* = c - p(t)$
2) $p(t) = k(t) + 1$
3) $c - 1 = |t\,|$
4) $|t\,| = |t \backslash t'| + |t \cap t'|$, for 'arbitrary tree t' in the graph.

Applying these equations to the pair of trees (t_1,t_2) we immediately obtain the relation:

$$\text{rank } t_1^* = |t_1 \backslash t_2| + |t_1 \cap t_2| - k(t_1)$$

Obviously the relation $\text{rank } t_1^* = |t_1 \backslash t_2|$ occurs iff $|t_1 \cap t_2| = k(t_1)$ □

Assertion 7

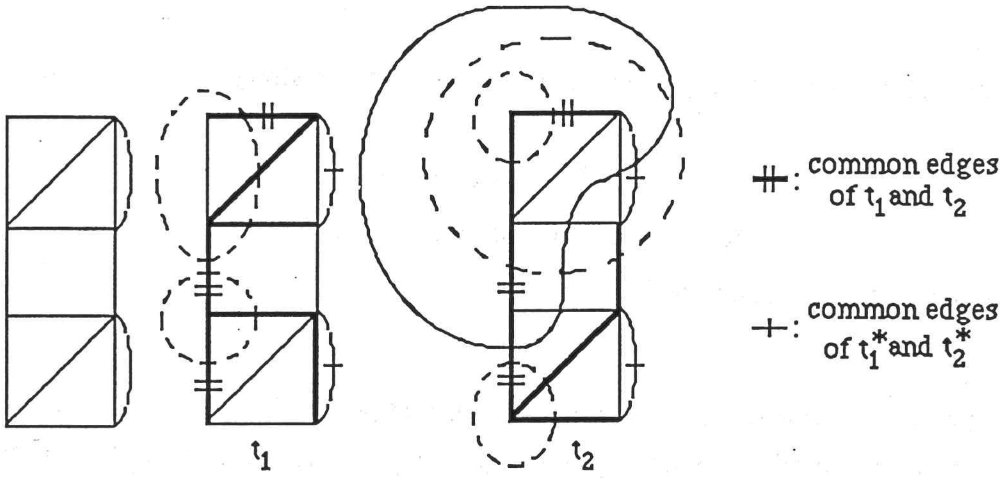
If t_2 is maximally distant from t_1, then each common edge of t_1 and t_2 belongs to a cutset of the graph which is made of edges in t_2 only.

Proof

If t_2 is maximally distant from t_1 than according to Assertion 6, part(iii), an edge in $t_1 \cap t_2$ can form a cutset only with the edges in $t_1 \backslash t_2$. But $t_1 \cap t_2$ and $t_1 \backslash t_2$ both belong to the tree t_2 and consequently that cutset is made of the edges in t_2 only. □

Remark 1

The converse to Assertion 7 is generally not true. To see this, consider the graph shown in figure 3(a) and the two trees shown in (b) and (c) of the same figure. Common edges are marked as indicated in (d). It can be seen by inspection that all common edges of t_1 and t_2 belong to cutsets t_2, that is, the condition of Assertion 7 is fulfilled. Nevertheless, there exists a fundamental cutset (indicated by the curvy solid line in (c)) with respect to t_2, defined by a common edge of t_1 and t_2 which includes a common edge of the cotrees t_1^* and t_2^*. According to Assertion 6, part iii), we conclude that t_2 is not maximally distant from t_1.



(a) (b) (c) (d)

Figure 3

The following version of Assertion 7 is also true: If t_2 is maximally distant from t_1, then each common edge of t_1^* and t_2^* belong to a circuit of the graph which is made of edges in t_1^* only.

§4 Perfect pairs of trees

A pair of trees (t_1,t_2) is defined to be a **perfect pair** of trees if both t_2 is maximally distant from t_1 and t_1 is maximally distant from t_2.

As an immediate consequence of Assertion 6 we have the following theorem that gives several equivalent characterisations of perfect pairs.

Theorem 1

The following five statements are equivalent:

i) (t_1,t_2) is a perfect pair

ii) fundamental circuits with respect to t_1 and t_2 defined by edges in $t_1^* \cap t_2^*$ contains no edges in $t_1 \cap t_2$

iii) fundamental cutsets with respect to t_1^* and t_2^* defined by edges in $t_1 \cap t_2$ contains no edges in $t_1^* \cap t_2^*$

iv) rank $t_1^* = |t_1 \backslash t_2| = |t_2 \backslash t_1| =$ rank t_2^*

v) the following three numbers, associated with the pair of trees (t_1,t_2) are equal:
- the maximal number of independent cutsets of the graph that belong to t_1
- the maximal number of independent cutsets of the graph, that belong to t_2
- the number of common edges in t_1 and t_2.

A pair (t_1,t_2) of trees is said to be a **maximally distant pair of trees** [3] if $|t_1\backslash t_2| \geq |t'\backslash t''|$ for every pair of trees (t', t'') in G.

Assertion 8

A maximally distant pair is a perfect pair.

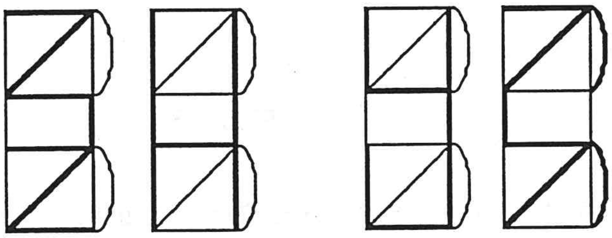
Proof

If pair (t_1, t_2) is maximally distant then $|t_1\backslash t_2| \geq |t'\backslash t''|$ for every pair (t',t'') in the graph. For the particular cases: $t'' = t_2$ and $t' = t_1$, we have $|t_1\backslash t_2| \geq |t_1\backslash t''|$ for every t'', and $|t_2\backslash t_1| \geq |t_2\backslash t'|$ for every t'.

Consequently, the pair (t_1, t_2) is a perfect pair. □

Remark 2

The converse of Assertion 8 is not generally true. Figure 4(a) shows a perfect pair of trees with a corresponding distance of 5. That this pair is not a maximally distant pair is easily seen by noting that figure 4(b) shows a pair with a corresponding distance of 7. This second pair is obviously a maximally distant pair because the intersection of their edge sets is empty.



(a) (b)

Figure 4

Remark 3

We point out to the reader that the application of theorem 1 to the domain of maximally distant pairs (which is a restriction of the domain of perfect pairs, see remark 2) leads to a restriction of the theorem in the following sense: conditions ii)-v) are only necessary but not sufficient. More precisely we have the following corollary.

Corollary (to Theorem 1 and Assertion 8)

If (t_1, t_2) is a maximally distant pair, then statements ii)-v) of theorem 1 hold.

A theorem related to maximally distant pairs similar to our corollary was presented in [21], (theorem 11.4). However, this theorem is only concerned with ii) and iii) of the corollary.

Assertion 9

If t_2 is maximally distant from t_1 then each edge in t_2 belongs to a fundamental circuit with respect to t_2, defined by an edge in $t_1\backslash t_2$.

Proof

Suppose that there exists an edge $x \in t_2$, that does not belong to a fundamental circuit with respect to

t_2 defined by edges in $t_1 \setminus t_2$. According to Assertion 4, such an edge as edge x can belong only to $t_1 \cap t_2$. On the other hand, in a 2-connected graph, each edge in t_2 must belong to some fundamental circuit with respect to t_2^*. This means that $x \in t_1 \cap t_2$ necessarily belongs to a fundamental circuit defined by an edge in $t_1^* \setminus t_2^*$, which contradicts the assumption that t_2 is maximally distant from t_1.

□

Remark 4

The converse statement to Assertion 9 is generally not true. To see this, consider the graph of figure 5(a) and the trees shown in (b) and (c) of that figure. Obviously rank $t_1^* = 3 >$ rank $t_2^* = 2 = |t_1 \setminus t_2|$. So, t_2 is not maximally distant from t_1. Nevertheless, each edge in t_2 belongs to a fundamental circuit with respect to t_2 defined by edges in $t_1 \setminus t_2$.

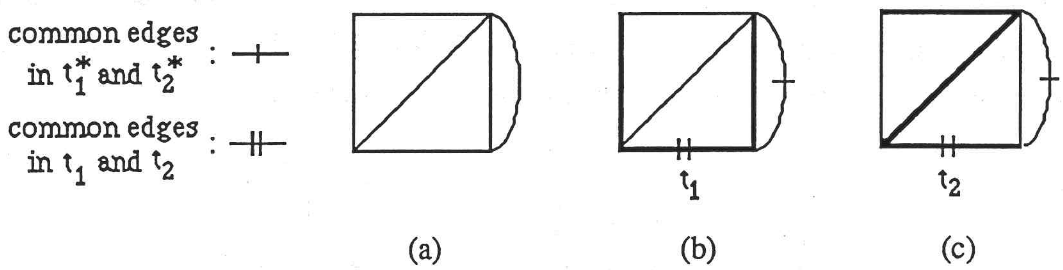


Figure 5

Denote by P the set of all distances between perfect pairs in the graph, and recall the definition of D.

Assertion 10

The following relation between P and D holds:

(i) $\min P \geq \min D$

(ii) $\max P = \max D$

Proof

(i) According to theorem 1, statement (iv), if t_1 and t_2 constitute a perfect pair, then they necessarily have the same diameter which is equal to their mutual distance apart. Hence, $D \supseteq P$ and consequently $\min P \geq \min D$.

(ii) The distance between a pair of maximally distant pair is equal to max D. But each maximally distant pair of trees is a perfect pair. Hence $\max P = \max D$. □

Remark 5

To prove the fact that strong inequality ($\min P > \min D$) can occur, let us consider the graph of figure 6(a). Analysing the maximum number of independent cutsets that do not contain circuits, it is easy to see that the diameter of the graph is 4. There are exactly five trees with diameter 4 and these are shown figure 6(b) (each of them includes five independent cutsets of the graph). It can be checked immediately that the distance between any two of them is equal to 2. Hence, there is no perfect pair with distance 4.

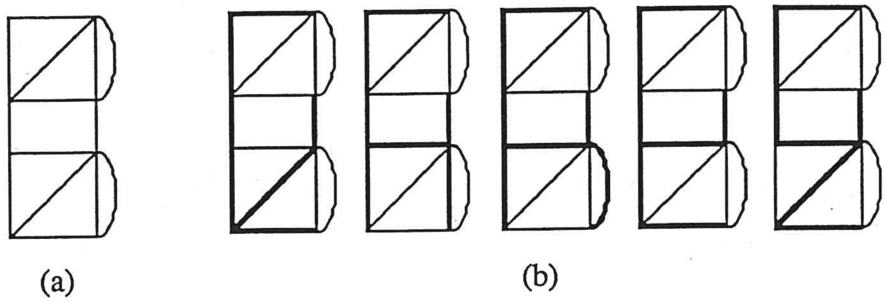


Figure 6

Conclusion

In this paper the new notion called a perfect pair of a graph was introduced. Several assertions were stated in order to closely characterise its properties. Also, several examples were included in order to help the reader gain intuitive insight.

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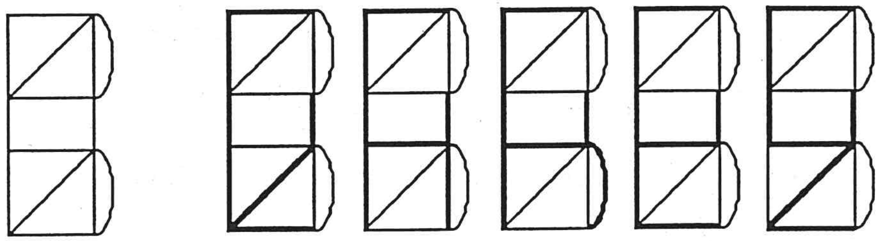

(a) (b)

Figure 6

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